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On continuity and boundedness property of PFNL operators

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Abstract: A more modern technique for addressing vulnerability is the picture fuzzy set (PFS). It is an instantaneous extension of intuitionistic fuzzy sets that can exhibit vulnerability in these situations, encompassing additional replies of the following types: yes, no, and decline. This study defines two varieties (strong and weak) of PFNL bounded linear operators and introduces the concept of boundedness of a linear operator from one PFNLS space to another PFNLS. Finally, the relationship between PFNL boundedness and PFNL continuity is examined.

Keywords: PFNLS; Norm; Co-norm; Continuity; Picture Fuzzy; Normed linear bounded operator

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1. Introduction

In [1], Prof. Zadeh introduced fuzzy set (FS) theory in 1965. To fight against unpre- dictability, FS was essentially the first step in generalizing the antiquated conception of classical set theory. However, FS also has limits. Atanassov [2, 3, 10] introduced the idea of Intuitionstic FS (IFS) theory and further generalized FS theory in order to get around the drawbacks. In this case, neutrality level was not taken into account. In their study [19], Prof. Cuong and Kreinovich introduced the idea of neutral membership into IFS theory and presented us with a lovely new set notion called Picture Fuzzy Set (PFS). It goes without saying that PFS is a generalization of IFS. As time went on, a number of researchers became interested in PFS theory and many kinds of research were conducted on it [9, 20, 22-25].

However, the foundation of functional analysis, a significant area of contemporary mathematics, is normed linear space. Fuzzy normed linear space (FNLS) was first presented by Prof. Felbin in 1992 [4]. [6, 11] shown that each finite dimensional FNLS has a unique fuzzy norm with respect to fuzzy equivalence. Prof. Felbin went on to prove in 1993 that any FNS with limited dimensions must be complete. Professor Bag and Samanta [12] broke down the fuzzy norm into a standard, crisp norm in 2003. This work showed the path for several methods of FN space research. Intuitionistic FNLS, Neutrosophic NLS, n-FNLS [5, 7, 8, 13-15, 21, 26, 28, 30] and other advances of FNLS have resulted from their publication. In 2024, we have first studied Picture Fuzzy normed linear space for the first time [31]. PFNLS is a far more generalized idea than the other one because PFS is a generalization of IFS. This article defines two varieties (strong and weak) of PFNL bounded linear operators and introduces the concept of boundedness of a linear operator from one PFNLS space to another PFNLS. Also, the relationship between PFNL boundedness and PFNL continuity is examined. Our manuscript is organized as following: In Section 2 some basics regarding PFNLS is given and a suitable example is provided for better understanding. In Section 3 continuity in PFNLS is shown. In the next section boundedness property of PFNLS is explored. Finally, Section 5 concludes our article.

2. Some basics on PFNLS

We ask that readers first read the concept of t-norm say $\odot$ and t-conorm say $\circ$ in order to realize PFNLS. One can find information about t-norm and t-conorm in any standard article, such as [26], because PFS is an ongoing evolution of FS and IFS. In 2013, Cuong *et al*. released the initial concept of PFS, a novel theory, in [19]. This section recapitulates the idea of PFNLS based on PFS theory which was discussed in [31]. Further we have provided an example of PFNLS for smooth understanding this article.

Definition 2.1. *[31] Suppose U be a linear space over R. A picture fuzzy subset*

$$A = \{((\alpha, r); P(\alpha, r), Q(\alpha, r), R(\alpha, r)) : (\alpha, r) \in U \times \mathbb{R}^{+}\}$$

is called a PF norm on U w. r. t. continuous t-norm $\odot$ and t-co-norm $\circ$ respectively if the following holds:

(a) $P(\alpha, r) + Q(\alpha, r) + R(\alpha, r) \leq 1 \ \forall (\alpha, r) \in U \times \mathbb{R}^{+}$.

(b) $P(\alpha, r) > 0$.

(c) $P(\alpha, r) = 1$ iff $\alpha = 0$.

(d) $P(k\alpha, r) = P(\alpha, r / |k|), k \in \mathbb{R} \setminus \{0\}$.

(e) $P(\alpha, r) \odot P(\beta, s) \leq P(\alpha + \beta, r + s)$.

(f) $P(\alpha, .)$ is non-decreasing mapping of R+ and $\lim_{r\to\infty} P(\alpha, r) = 1$.

(g) $Q(\alpha, r) > 0$.

(h) $Q(\alpha, r) = 0$ if and only if $\alpha = 0$.

(i) $Q(k\alpha, r) = Q(\alpha, r), k \in R \setminus \{0\}$.

(j) $Q(\alpha, r) \circ Q(\beta, s) \geq Q(\alpha + \beta, r + s)$.

(k) $Q(\alpha, .)$ is non-increasing function of R+ and $\lim_{r\to\infty} Q(\alpha, r) = 0$.

(l) $R(\alpha, r) > 0$.

(m) $R(\alpha, r) = 0$ iff $x = 0$.

(n) $R(k\alpha, r) = R(\alpha, r), k \in R \setminus \{0\}$.

(o) $R(\alpha, r) \circ R(\beta, s) \geq R(\alpha + \beta, r + s)$.

(p) $R(\alpha, .)$ is non-increasing mapping of R+ and $\lim_{r\to\infty} R(\alpha, r) = 0$.

Here $(U, A, \odot, \circ)$ is called a PFNLS. We will denote $(U, A, \odot, \circ)$ as (U, A) throughout this article.

Example 2.2. *Suppose $U = (\mathbb{R}, \|.\|)$ be NLS with $\|.\| = |x| \ \forall \ x \in \mathbb{R}$. Considering $a_1 \odot a_2 = \min\{a_1, a_2\}$ and $a_1 \circ a_2 = \max\{a_1, a_2\} \ \forall \ a_1, a_2 \in [0, 1]$ we take $P(\alpha, r) = \frac{r}{r+c|\alpha|}, Q(\alpha, r) = \frac{c|\alpha|}{r+c|\alpha|}, R(\alpha, r) = \frac{|\alpha|}{r}, \ c > 0$. We take $A = \{(\alpha, r); P(\alpha, r), Q(\alpha, r), R(\alpha, r)\}$. Clearly $(U, A, \odot, \circ)$ is an PFNLS.*

3. Continuity in PFNLS

In this section we will study various types of continuity of an operator over PFNLS.

Definition 3.1. *Suppose $(W, A_W, \odot, \circ)$ and $(Z, A_Z, \odot, \circ)$ be two PFNLSs. A mapping $\mathfrak{T} : (W, A_W, \odot, \circ) \to (Z, A_Z, \odot, \circ)$ is said to be PF continuous at $w_0 \in W$ if for all $w \in W$ for each $\mu > 0, m \in (0, 1), \ \exists \ \nu > 0, n \in (0, 1)$ s.t.*

$$P_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), m) > (1 - \mu), \ Q_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), m) < \mu, \ R_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), m) < \mu$$

whenever

$$P_W(w - w_0), n) > (1 - \nu), \ Q_W(w - w_0), n) < \nu, \ R_W(w - w_0), n) < \nu$$

Definition 3.2. *Suppose $(W, A_W, \odot, \circ)$ and $(Z, A_Z, \odot, \circ)$ be two PFNLSs. A mapping $\mathfrak{T} : (W, A_W, \odot, \circ) \to (Z, A_Z, \odot, \circ)$ is said to be*

- *strongly PF continuous at $w_0 \in W$ if for all $w \in W$ for each $\mu > 0 \ \exists \ \nu > 0$ s.t.*

$$P_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), \mu) \geq P_W(w - w_0), \nu),$$
$$Q_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), \mu) \leq Q_W(w - w_0), \nu),$$
$$R_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), \mu) \leq R_W(w - w_0), \nu).$$

- *weakly PF continuous at $w_0 \in W$ if for all $w \in W$ for each $\mu > 0, \ m \in (0, 1) \ \exists \ \nu > 0$ s.t.*

$$P_W(w - w_0), \nu) \geq m \to P_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), \mu) \geq m$$
$$Q_W(w - w_0), \nu) \leq m \to Q_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), \mu) \leq m$$
$$R_W(w - w_0), \nu) \leq m \to R_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), \mu) \leq m$$

Definition 3.3. *Suppose $(W, A_W, \odot, \circ)$ and $(Z, A_Z, \odot, \circ)$ be two PFNLSs. A mapping $\mathfrak{T} : (W, A_W, \odot, \circ) \to (Z, A_Z, \odot, \circ)$ is said to be sequentially PF continuous at $w_0 \in W$ if for any sequence $\{w_n\}$ in W satisfying $w_n \to w_0$ implies that $\mathfrak{T}(w_n) \to \mathfrak{T}(w_0)$ i.e.*

$$\lim_{n\to\infty} P_W(w_n - w_0), m) = 1 \to \lim_{n\to\infty} P_Z(\mathfrak{T}(w_n) - \mathfrak{T}(w_0), m) = 1$$
$$\lim_{n\to\infty} Q_W(w_n - w_0), m) = 0 \to \lim_{n\to\infty} Q_Z(\mathfrak{T}(w_n) - \mathfrak{T}(w_0), m) = 0$$
$$\lim_{n\to\infty} R_W(w_n - w_0), m) = 0 \to \lim_{n\to\infty} R_Z(\mathfrak{T}(w_n) - \mathfrak{T}(w_0), m) = 0,$$

where $m > 0$.

Theorem 3.4. *Suppose $(W, A_W, \odot, \circ)$ and $(Z, A_Z, \odot, \circ)$ be two PFNLSs and $\mathfrak{T} : (W, A_W, \odot, \circ) \to (Z, A_Z, \odot, \circ)$ be mapping. Then strong continuity of $\mathfrak{T}$ implies sequentially continuity of $\mathfrak{T}$ at a point $w_0 \in W$.*

Proof. Suppose $\{w_n\}$ be a convergent sequence converging to $w_0 \in W$. Since $\mathfrak{T}$ is strongly continuous then we have for all $w \in W$, for each $\mu > 0 \, \exists \, \nu > 0$ s.t.

$$\begin{aligned} P_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), \mu) &\geq P_W(w - w_0), \nu), \\ Q_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), \mu) &\leq Q_W(w - w_0), \nu), \\ R_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), \mu) &\leq R_W(w - w_0), \nu). \end{aligned}$$

Now by putting $w = w_n$ we have

$$\begin{aligned} P_Z(\mathfrak{T}(w_n) - \mathfrak{T}(w_0), \mu) &\geq P_W(w_n - w_0), \nu), \\ Q_Z(\mathfrak{T}(w_n) - \mathfrak{T}(w_0), \mu) &\leq Q_W(w_n - w_0), \nu), \\ R_Z(\mathfrak{T}(w_n) - \mathfrak{T}(w_0), \mu) &\leq R_W(w_n - w_0), \nu). \end{aligned}$$

which clearly implies that $\mathfrak{T}(w_n) \to \mathfrak{T}(w_0)$ i.e. $\mathfrak{T}$ is sequentially convergent. ∎

However the converse is not true i.e. sequential continuity in a PFNLS does not imply strong continuity. Our following example demonstrates this.

Example 3.5. *Suppose $S = \langle \mathbb{R}, ||x||, \odot, \circ \rangle$ be a PFNLS, where $||x|| = |x|, a \odot b = \min\{a, b\}, a \circ b = \max\{a, b\} \ \forall \ a, b \in [0, 1]$. We define the functions $P_i, Q_i, R_i, i = 1, 2 : S \times \mathbb{R}^+ \to [0, 1]$ by*

$$\begin{aligned} P_1(x, r) = \frac{r}{r + |x|}, \ P_2(x, r) &= \frac{r}{r + w|x|}, w > 0 \\ Q_1(x, r) = \frac{|x|}{r + |x|}, \ Q_2(x, r) &= \frac{|x|}{r + w|x|}, w > 0 \\ R_1(x, r) = \frac{|x|}{r}, \ P_2(x, r) &= \frac{|x|}{wr}, w > 0 \end{aligned}$$

It is clear that $\langle S, A_{S_i}, \odot, \circ \rangle$, $i = 1, 2$ are PFNLSs. Suppose $g : \langle S, A_{S_1}, \odot, \circ \rangle \to \langle S, A_{S_2}, \odot, \circ \rangle$ be a mapping defined as $g(x) = \frac{x^5}{1+x^2} \ \forall \ x \in U$. Suppose $s_0 \in S$ and $\{s_n\}$ be a sequence in S such that $s_n \to s_0$ in $\langle S, A_{S_1}, \odot, \circ \rangle$. Then one can easily verify that $\forall \ r > 0$,

$$\lim_{n \to \infty} P_1(s_n - s_0, r) = 1, \lim_{n \to \infty} Q_1(s_n - s_0, r) = 0, \lim_{n \to \infty} R_1(s_n - s_0, r) = 0$$

Now we calculate $P_2(g(s_n) - g(s_0), r), Q_2(g(s_n) - g(s_0), r), R_2(g(s_n) - g(s_0), r)$. In every terms of $Q_2(g(s_n) - g(s_0), r), R_2(g(s_n) - g(s_0), r)$ consists a term $(s_n - s_0)$ in numerator which becomes zero as n tends to ∞. Thus $\lim_{n \to \infty} Q_2(g(s_n) - g(s_0), r) = 0$, $\lim_{n \to \infty} R_2(g(s_n) - g(s_0), r) = 0$. In a parallel manner, it is seen that $\exists$ also a term $(s_n - s_0)$ in denominator in $P_2(g(s_n) - g(s_0), r)$. As a result we get $\lim_{n \to \infty} P_2(g(s_n) - g(s_0), r) = 1$ and hence g is sequentially convergent on S. On the other hand, assume g is strongly continuous in S

i.e. $\forall\, s_0 \in S$ and for each $r > 0$ such that for all $s_0 \in U$,

$$\begin{aligned} P_1(s - s_0, t) &\leq P_2((g(s_n) - g(s_0), r), \\ Q_1(s - s_0, t) &\geq Q_2((g(s_n) - g(s_0), r), \\ R_1(s - s_0, t) &\geq Q_2((g(s_n) - g(s_0), r). \end{aligned}$$

Now

$$\begin{aligned} P_2((g(s_n) - g(s_0), r) &\geq P_1(s - s_0, t) \\ \Longrightarrow \frac{r}{r + w\left|\frac{s^5}{1+s^2} - \frac{s^5}{1+s^2}\right|} &\geq \frac{t}{t + |s - s_0|} \\ \Longrightarrow t &\leq \frac{r}{w} h\left(\frac{1}{s}, s_0\right) \\ \Longrightarrow \inf_{s \in S} h\left(\frac{1}{s}, s_0\right) &\geq \frac{wt}{r}, \end{aligned}$$

where h is a polynomial in s, $s \neq s_0$ with degree ≤ 1. Choose $t_1 = \inf |h(\frac{1}{s})|$. Thus $\frac{wt}{r} = 0$. Since $w, r > 0$, then it implies that $t = 0$ which gives a contradiction to the fact $t > 0$. So, g is not strongly continuous.

Theorem 3.6. *Suppose* $(W, A_W, \odot, \circ)$ *and* $(Z, A_Z, \odot, \circ)$ *be two PFNLSs and* $\mathfrak{T} : (W, A_W, \odot, \circ) \rightarrow (Z, A_Z, \odot, \circ)$ *be mapping. Then continuity of* $\mathfrak{T}$ *implies sequentially continuity of* $\mathfrak{T}$ *and vice-versa at a point* $w_0 \in W$.

Proof. Suppose that $\mathfrak{T}$ is continuous at $w_0 \in W$ and $\{w_n\}$ is a convergent sequence that converges to w_0 in W. Then for all $w \in W$, for each $\mu \in (0, 1)$ and $r > 0$, $\exists\, \nu \in (0, 1)$ and $m > 0$ such that,

$$\begin{aligned} P_W(w - w_0, m) > (1 - \nu) &\Rightarrow P_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), r) > (1 - \mu), \\ Q_W(w - w_0, m) < \nu &\Rightarrow Q_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), r) < \mu, \\ R_W(w - w_0, m) < \nu &\Rightarrow R_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), r) < \mu \end{aligned}$$

Since $\{w_n\} \rightarrow w_0$, thus $\exists\, n_0 \in \mathbb{N}$ such that,

$$P_W(w - w_0, m) > (1 - \nu), Q_W(w - w_0, m) < \nu, R_W(w - w_0, m) < \nu$$

Hence,

$$\begin{aligned} P_Z(\mathfrak{T}(w_n) - \mathfrak{T}(w_0), r) &> (1 - \mu), \\ Q_Z(\mathfrak{T}(w_n) - \mathfrak{T}(w_0), r) &< \mu, \\ R_Z(\mathfrak{T}(w_n) - \mathfrak{T}(w_0), r) &< \mu \end{aligned}$$

which clearly shows that $\mathfrak{T}$ is sequentially continuous at $w_0 \in W$. Conversely suppose that $\mathfrak{T}$ is sequentially continuous at $w_0 \in W$ and if possible $\mathfrak{T}$ is not continuous at w_0 i.e. $\exists\, \mu \in (0, 1)$ and $r > 0$ such that for any $\nu \in (0, 1)$ and $m > 0$ such that

$$\begin{aligned} P_W(w - w_0, m) > (1 - \nu) &\Rightarrow P_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), r) \leq (1 - \mu), \\ Q_W(w - w_0, m) < \nu &\Rightarrow Q_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), r) \geq \mu, \\ R_W(w - w_0, m) < \nu &\Rightarrow R_Z(\mathfrak{T}(w) - \mathfrak{T}(w_0), r) \geq \mu \end{aligned}$$

So, for $\nu = 1 - \frac{1}{1+p}, m = \frac{1}{p+1},\ p \in \mathbb{N}, \exists\, w_p$ such that

$$P_W(w_p - w_0, \frac{1}{p+1}) > \frac{1}{p+1} \text{ but } P_Z(\mathfrak{T}(w_p) - \mathfrak{T}(w_0), r) \leq (1-\mu),$$
$$Q_W(w - w_0, \frac{1}{p+1}) < 1 - \frac{1}{p+1} \text{ but } Q_Z(\mathfrak{T}(w_p) - \mathfrak{T}(w_0), r) \geq \mu,$$
$$R_W(w - w_0, \frac{1}{p+1}) < 1 - \frac{1}{p+1} \text{ but } R_Z(\mathfrak{T}(w_p) - \mathfrak{T}(w_0), r) \geq \mu.$$

Considering $m > 0,\ \exists\, p_0$ s.t. $\frac{1}{p+1} < m\ \forall\, p \geq p_0$, we have

$$\lim_{n\to\infty} P_W(w_p - w_0, m) = 1;$$
$$\lim_{n\to\infty} Q_W(w_p - w_0, m) = 0,$$
$$\lim_{n\to\infty} R_W(w_p - w_0, m) = 0$$

this lead to $w_p \to w_0$, although

$$\begin{aligned} P_Z(\mathfrak{T}(w_p) - \mathfrak{T}(w_0), r) &\leq (1-\mu), \\ Q_Z(\mathfrak{T}(w_p) - \mathfrak{T}(w_0), r) &\geq \mu, \\ R_Z(\mathfrak{T}(w_p) - \mathfrak{T}(w_0), r) &\geq \mu. \end{aligned}$$

i.e. $\mathfrak{T}(w_p)$ does not converge $\mathfrak{T}(w_0)$ which gives a contradiction. Therefore, the result follows. ■

4. PFN bounded operator

Throughout this section, we will discuss the idea of boundedness and isometry property between PFN linear operators between PFNLSs. Also we will study some relationships between different type of operators.

Definition 4.1. *Suppose $(W, A_W, \odot, \circ)$ and $(Z, A_Z, \odot, \circ)$ are two PFNLSs and $\mathfrak{T} : (W, A_W, \odot, \circ) \to (Z, A_Z, \odot, \circ)$ is a linear operator. Then $\mathfrak{T}$ is called strongly PFN bounded if there exists a non-zero constant real number r such that for each $w \in W$ and $\forall\, s > 0$,*

$$\begin{aligned} P_Z(\mathfrak{T}(w), s) &\geq P_W(rw, s), \\ Q_Z(\mathfrak{T}(w), s) &\leq Q_W(rw, s), \\ R_Z(\mathfrak{T}(w), s) &\leq R_W(rw, s). \end{aligned}$$

It is quite clear that the zero operators and and identity operators are by definition a strongly PFN bounded operator.

Example 4.2. *Suppose $U = (\mathbb{R}, \|.\|)$ be NLS with $\|.\| = |x|\ \forall\ x \in \mathbb{R}$. Considering $a_1 \odot a_2 = \min\{a_1, a_2\}$ and $a_1 \circ a_2 = \max\{a_1, a_2\}\ \forall\ a_1, a_2 \in [0,1]$ we take $P(\alpha, t) = \frac{t}{t+cm_1|\alpha|}, Q(\alpha, t) = \frac{c|\alpha|}{t+c|\alpha|}, R(\alpha, t) = \frac{|\alpha|}{t},\ c > 0,\ m_1$ is fixed real number. We take $A = \{(\alpha, t); P(\alpha, t), Q(\alpha, t), R(\alpha, t)\}$. Clearly $(U, A, \odot, \circ)$ is an PFNLS. Now we choose $P_1(\alpha, t) = \frac{t}{t+cm_2|\alpha|}, Q_1(\alpha, t) = \frac{c|\alpha|}{t+c|\alpha|}, R_1(\alpha, t) = \frac{|\alpha|}{t},\ c > 0,\ m_2$ is fixed real number and $m_1 > m_2$. We take $B = \{(\alpha, t); P_1(\alpha, t), Q_1(\alpha, t), R_1(\alpha, t)\}$. Clearly $(U, B, \odot, \circ)$ is also* *a PFNLS. Now we consider an linear operator* $\mathfrak{T} : (U, A, \odot, \circ) \to (U, B, \odot, \circ)$ *given by* $\mathfrak{T}(x) = rx,\ r \in \mathbb{R}\ \{0\}$. *Clearly* $\mathfrak{T}$ *is a strongly PFN bounded operator.*

Definition 4.3. *Suppose* $(W, A_W, \odot, \circ)$ *and* $(Z, A_Z, \odot, \circ)$ *are two PFNLSs and* $\mathfrak{T} : (W, A_W, \odot, \circ) \to (Z, A_Z, \odot, \circ)$ *is a linear operator. Then* $\mathfrak{T}$ *is called weakly PFN bounded if for* $\mu \in (0, 1)$ *there exists a non zero constant real number* r *such that for each* $w \in W$ *and* $s > 0$,

$$\begin{aligned} P_Z(rw, s) \geq 1 - \mu \quad &\Rightarrow \quad P_W(\mathfrak{T}(w), s) \geq 1 - \mu, \\ Q_Z(rw, s) \leq \mu \quad &\Rightarrow \quad Q_W(\mathfrak{T}(w), s) \leq \mu, \\ R_Z(rw, s) \leq \mu \quad &\Rightarrow \quad R_W(\mathfrak{T}(w), s) \leq \mu, \end{aligned}$$

Definition 4.4. *Suppose* $(W, A_W, \odot, \circ)$ *and* $(Z, A_Z, \odot, \circ)$ *are two PFNLSs and* $\mathfrak{T} : (W, A_W, \odot, \circ) \to (Z, A_Z, \odot, \circ)$ *is a linear operator. Then* $\mathfrak{T} : W \to Z$ *is called PFNL isometry if for each* $w \in W, s > 0$ *such that*

$$\begin{aligned} P_Z(\mathfrak{T}(w), s) &= P_W(w, s), \\ Q_Z(\mathfrak{T}(w), s) &= Q_W(w, s), \\ R_Z(\mathfrak{T}(w), s) &= R_W(w, s). \end{aligned}$$

Theorem 4.5. *Suppose* $(W, A_W, \odot, \circ)$ *and* $(Z, A_Z, \odot, \circ)$ *are two PFNLSs and* $\mathfrak{T} : (W, A_W, \odot, \circ) \to (Z, A_Z, \odot, \circ)$ *is a linear operator. If* $\mathfrak{T}$ *is PFN strongly bounded, then it is PFN weakly bounded.*

Proof. Suppose $\mathfrak{T}$ is PFN strongly bounded. Then there exists a nonzero constant real number r such that for each $w \in W$ and $\forall\ s > 0$,

$$\begin{aligned} P_Z(\mathfrak{T}(w), s) &\geq P_W(rw, s), \\ Q_Z(\mathfrak{T}(w), s) &\leq Q_W(rw, s), \\ R_Z(\mathfrak{T}(w), s) &\leq R_W(rw, s). \end{aligned}$$

Since $P_W(rw, s), Q_W(rw, s), R_W(rw, s) \in [0, 1]$, we obtain that for any $0 < \mu < 1$ s.t.

$$\begin{aligned} P_Z(\mathfrak{T}(w), s) &\geq P_W(rw, s) \geq 1 - \mu, \\ Q_Z(\mathfrak{T}(w), s) &\leq Q_W(rw, s) \leq \mu, \\ R_Z(\mathfrak{T}(w), s) &\leq R_W(rw, s) \leq \mu. \end{aligned}$$

Clearly $\mathfrak{T}$ is PFN weakly bounded. ∎

Theorem 4.6. *Suppose* $(W, A_W, \odot, \circ)$ *and* $(Z, A_Z, \odot, \circ)$ *are two PFNLSs and* $\mathfrak{T} : (W, A_W, \odot, \circ) \to (Z, A_Z, \odot, \circ)$ *is a linear operator. If* $\mathfrak{T}$ *is PFN strongly bounded if and only if* $\mathfrak{T}$ *is continuous.*

Proof. Suppose $\mathfrak{T}$ is PFN strongly bounded. Then for a nonzero real constant r such that for every $s > 0$, we have

$$\begin{aligned} P_Z(\mathfrak{T}(w), s) &\geq P_W(rw, s) = P_W(w, \frac{s}{|r|}) = P_W(w, t), \\ Q_Z(\mathfrak{T}(w), s) &\leq Q_W(rw, s) = Q_W(w, \frac{s}{|r|}) = Q_W(w, t), \end{aligned}$$

$$R_Z(\mathfrak{T}(w), s) \leq R_W(rw, s) = R_W(w, \frac{s}{|r|}) = R_W(w, t).$$

Suppose $w_0 \in W, \mu \in (0,1), s > 0$. We put $\mu = \nu$ and $t = \frac{r}{k} > 0$. Also suppose that

$$P_W(w - w_0) \geq (1-\mu),\ Q_W(w - w_0) \leq \mu,\ R_W(w - w_0) \leq \mu.$$

Then clearly we have,

$$\begin{aligned} P_Z((\mathfrak{T}(w) - (\mathfrak{T}(w_0)) &\geq 1 - \mu, \\ Q_Z((\mathfrak{T}(w) - (\mathfrak{T}(w_0)) &\leq \mu, \\ R_Z((\mathfrak{T}(w) - (\mathfrak{T}(w_0)) &\leq \mu. \end{aligned}$$

Hence $\mathfrak{T}$ is continuous. On the other hand, let us assume that $\mathfrak{T}$ is continuous on $(W, A_W, \odot, \circ)$. Then from the definition of continuity we have $\forall\ w \in W, 0 < \mu < 1$ and $r > 0, \exists\ 0 < \nu < 1$ and $t > 0$ we have,

$$\begin{aligned} P_W(w - w_0) \geq (1-\mu) \Rightarrow P_Z((\mathfrak{T}(w) - (\mathfrak{T}(w_0)) &\geq 1 - \mu \\ Q_W(w - w_0) \leq \mu \Rightarrow Q_Z((\mathfrak{T}(w) - (\mathfrak{T}(w_0)) &\leq \mu \\ R_W(w - w_0) \leq \mu \Rightarrow R_Z((\mathfrak{T}(w) - (\mathfrak{T}(w_0)) &\leq \mu \end{aligned}$$

Now for any $k \in (0,1)$ such that,

$$P_W(kw) \geq (1-\nu), Q_W(kw) \leq \mu, R_W(kw) \leq \mu.$$

Thus,

$$\begin{aligned} P_W(w, \frac{r}{|k|}) &= P_W(w, r) \geq (1-\nu), \\ Q_W(w, \frac{r}{|k|}) &= Q_W(w, r) \leq \nu \\ R_W(w, \frac{r}{|k|}) &= R_W(w, r) \leq \nu \end{aligned}$$

By assuming $t = \frac{r}{k}$, we obtain that

$$\begin{aligned} P_W(w, t) \geq (1-\mu) \Rightarrow P_Z((\mathfrak{T}(w), r) &\geq 1 - \mu \\ Q_W(w, t) \leq \mu \Rightarrow Q_Z(\mathfrak{T}(w), r) &\leq \mu \\ R_W(w, t) \leq \mu \Rightarrow R_Z((\mathfrak{T}(w), r) &\leq \mu \end{aligned}$$

■

Hence the result follows.

5. Conclusion

In this article, the continuity and boundedness property of PFNL operators are dis- cussed. Further some properties related to continuous and bounded operators are discussed. All these concepts are illustrated with examples. Also, we have studied some interested relationship theorem between the continuity and boundedness of a operator on PFNLS. These results are partially not true for the existing fuzzy set structures. In the future, we will study the concept of PF n-NLS and PF inner product space.

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